Dynamics of Viscoelastic Structures

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In this paper we present a general method for modeling material damping in dynamical systems. The work is primarily concerned with a dissipation model based on viscoelastic assumptions. Motion equations are formulated in operator form for a structure constructed from an anisotropic, viscoelastic material. The mass, damping, and stiffness operators are developed consistently in the formulation. Basic operator properties are discussed, and orthonormality conditions are derived for the viscoelastic system. Modal identities are derived for a constrained viscoelastic structure. These identities provide useful criteria in order reduction of finite-element models. An example of a viscoelastic beam in pure flexure is illustrated.

I. Introduction

THE analysis of most structural dynamics problems can be undertaken with considerable confidence owing to the powerful numerical and analytical techniques currently available. Damping models are, however, usually the weakest link in a structural dynamics analysis. Hence, the primary objective of our work is to present a consistent material damping model. A number of techniques exist for identifying the damping matrix given a set of information about the dynamic behavior of the system. For example, Caravani et al.[1] have outlined a time domain recursive least squares technique to identify the general damping matrix. Similarly, Yun and Shinozuka[2] describe a nonlinear Kalman filtering technique for this problem. Another method as given by Caravani and Thompson[3] of characterizing systems with nonproportional damping is to work in the frequency domain. Although these techniques are capable of determining the system damping matrix, the drawback to identifying this matrix from a given set of information about the dynamic behavior of the system is that the damping matrix is not known at the design stage. Also, alterations in the parameters of the system require a remeasurement of the dynamic behavior to identify the damping matrix by measuring the dynamic response may be impractical for the altered system. Of course, identifying the damping matrix by measuring the dynamic response may be impractical for some structures, such as large, flexible space structures. The current approach to the problem is to construct a consistent damping matrix without any knowledge of the dynamic behavior of the system.

In the current work, the underlying assumption in deriving the motion equation is that the stress-strain and the strain-displacement relations behave linearly and the material is viscoelastic. Otherwise, the material damping model is quite general and can be applied to any material with an arbitrary degree of anisotropy. The dynamics of an arbitrary structure is studied by formulating the motion equation using an operator notation.

In Sec. II, we briefly review Biot's viscoelastic stress-strain relation. We then use this in Sec. III to develop the motion equation. The assumption of linearity permits us to formulate the eigenproblem by taking the Laplace transform. Orthonormality conditions are the topic of discussion of Sec. VI: first, the elastic problem is reviewed, and the remainder of the section is devoted to the orthonormality conditions of the viscoelastic problem. The previous results are the basis for the derivation of modal identities; these provide a useful guide for establishing a hierarchy of system modes. Finally, the general results are illustrated in a cantilevered viscoelastic beam problem.

II. Some Aspects of Linear Viscoelasticity

Structural models based on purely elastic constitutive relations cannot account for energy dissipation. For a comprehensive development, we require constitutive relations that allow the time rate of change of energy to be negative. This condition is met by viscoelastic constitutive relations.

Viscoelastic constitutive relations of an anisotropic material can be given in terms of the Boltzmann hereditary superposition integral where the stress depends on the entire strain history:

$$\sigma_{ij}(t)=\int_0^t \sum_{k,l=1}^{3} C_{ij}^{kl}(r,t-\tau)\frac{\partial \epsilon_{kl}}{\partial \tau}\,\mathrm{d}\tau \quad i,j=1,2,3 \tag{1}$$

Here, the usual tensor notation followed by Sokolnikoff[4] is used. The tensor quantities refer to a dextral orthogonal set of Cartesian axes (x_1,x_2,x_3); r is the position vector; C_{ij}^{kl}, which is a function of r and t, is termed the relaxation moduli; σ_{ij} is the stress tensor; and ϵ_{ij} is the strain tensor.

The time dependence of the relaxation moduli is taken as a series of exponentials (Staverman and Schwarz[15])

$$C_{ij}^{kl}(r,t)=\sum_{s=1}^{n} C_{ij}^{kl(s)}(r)e^{-t/\rho_s}+C_{ij}^{kl}(r) \tag{2}$$

which is the harbinger of the more general relation to follow. The exponential series in Eq. (2) implies that the stress-strain relation can be represented by an array of springs and dashpots.[6] In Eq. (2), ρ_s are the material relaxation (decay) constants, and all fourth-order tensors on the right-hand side are independent of t, completely symmetric, real, and positive-semidefinite.

A possibly large number of moduli C_{ij}^{kl} and relaxation constants ρ_s in Eq. (2) may be required to describe a viscoelastic material. To circumvent this dilemma, Biot[7] considered an almost continuous distribution of relaxation constants. Hence, the discrete relaxation constants are treated in the form of a continuous relaxation density distribution function $\gamma(\theta)$. Therefore, the viscoelastic constitutive relation takes the form

$$\sigma_{ij}(t)=\int_0^t \sum_{k,l=1}^{3} \left[G_{ij}^{kl}(r,t-\tau)+C_{ij}^{kl}(r)\right]\frac{\partial \epsilon_{kl}}{\partial \tau}\,\mathrm{d}\tau \tag{3}$$

$$G_{ij}^{kl}(r,t)=\int_0^\infty C_{ij}^{kl(\cdot)}(r,\theta)\gamma(\theta)e^{-\theta t}\mathrm{d}\theta \tag{4}$$

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where G_{ij}^{kl} are completely symmetric and positive-semi-definite, $C_{ij}^{kl(\cdot)}$ is a function of r, and θ is the continuous distribution of the reciprocal of the material relaxation constants ρ_s. The summation in Eq. (2) is replaced by an integral in Eq. (4), accounting for the possibility of $C_{ij}^{kl}(r,\theta)$ being a discontinuous function in θ.

It will be convenient in our formulation to separate the elastic term; hence, we integrate the second component (time-independent moduli) in Eq. (3) to obtain

$$\sigma_{ij}(t)=\int_0^t \sum_{k,l=1}^{3} G_{ij}^{kl}(r,t-\tau)\frac{\partial \epsilon_{kl}}{\partial \tau}\,\mathrm{d}\tau+\sum_{k,l=1}^{3} C_{ij}^{kl}(r)\epsilon_{kl} \tag{5}$$

This relation will be the building block in our development of the motion equations of a viscoelastic body. We note that for $G_{ij}^{kl}\equiv 0$, the linear elastic anisotropic constitutive equation is recovered.

III. Theory

We are now in a position to formulate the motion equations of a viscoelastic body. The motion equations for a viscoelastic continuum are developed in operator form. In so doing we apply the principle of virtual work, which can be stated as follows: It is a necessary and sufficient condition for system equilibrium if the total work done by all the external forces plus the work done by all the internal forces is equal to zero for an arbitrary virtual displacement.

This principle can be generalized to include dynamical systems by appealing to d'Alembert's principle. Hence, the principle of virtual work can be stated as

$$\int_V \sum_{i=1}^{3} f_i\delta u_i \mathrm{d}V=\int_V \sum_{i=1}^{3}\left(\sum_{j=1}^{3}\sigma_{ij}\delta\epsilon_{ij}+\dot{p}_i\delta u_i\right)\mathrm{d}V \tag{6}$$

where f_i are the components of the externally applied force vector

$$f^T=[f_1(r)f_2(r)f_3(r)] \tag{7}$$

taken as force per unit volume; p_i are the components of the momentum vector

$$p^T=[p_1(r)p_2(r)p_3(r)] \tag{8}$$

and are defined as

$$P_i\triangleq\rho(r)\dot{u}_i \tag{9}$$

$$(\dot{\ }) \triangleq \frac{\partial}{\partial t} \tag{10}$$

where $\rho(r)$ is the material density of the viscoelastic continuum, taken to be time-independent but in general nonuniform. The linear strain tensor is

$$\epsilon_{ij}\triangleq \tfrac{1}{2}(u_{i,j}+u_{j,i}) \tag{11}$$

where u_i are components of the displacement vector

$$u^T=[u_1(r,t)\ u_2(r,t)\ u_3(r,t)] \tag{12}$$

We have used δu_i in Eq. (6) to denote the components of virtual displacements in contrast to the components of real displacements u_i.

Using Eqs. (5) and (11) in Eq. (6) gives

$$\int_V \sum_{i=1}^{3} f_i\delta u_i\mathrm{d}V=\int_V \sum_{i=1}^{3}\Bigg\{\frac{1}{4}\sum_{j=1}^{3}\Bigg[\int_0^t \sum_{k,l=1}^{3} G_{ij}^{kl}(r,t-\tau) \times\frac{\partial}{\partial\tau}(u_{k,l}+u_{l,k})\mathrm{d}\tau+\sum_{k,l=1}^{3} C_{ij}^{kl}(r)(u_{k,l}+u_{l,k})\Bigg] \times\delta(u_{i,j}+u_{j,i})+\dot{p}_i\delta u_i\Bigg\}\mathrm{d}V \tag{13}$$

Applying the divergence theorem to the volume integral and making use of the symmetry properties of moduli G_{ij}^{kl} and C_{ij}^{kl} gives

$$\int_V \sum_{i=1}^{3}\Bigg\{\dot{p}_i-\sum_{j,k,l=1}^{3}\frac{\partial}{\partial x_j}\Bigg[\int_0^t G_{ij}^{kl}(r,t-\tau)\frac{\partial u_{l,k}}{\partial\tau}\mathrm{d}\tau +C_{ij}^{kl}(r)u_{l,k}\Bigg]-f_i\Bigg\}\delta u_i\mathrm{d}V=0 \tag{14}$$

with the appropriate boundary terms compatible with the boundary conditions of the problem. Since the components of the virtual displacements are arbitrary and independent of one another, it follows that

$$\dot{p}_i-\sum_{j,k,l=1}^{3}\frac{\partial}{\partial x_j}\Bigg[\int_0^t G_{ij}^{kl}(r,t-\tau)\frac{\partial u_{l,k}}{\partial\tau}\mathrm{d}\tau +C_{ij}^{kl}(r)u_{l,k}\Bigg]-f_i=0,\qquad i=1,2,3 \tag{15}$$

Therefore, in operator form, the motion equation can be written as an integro-differential equation

$$\underset{\sim}{M}\ddot{u}+\int_0^t \underset{\sim}{R}(r,t-\tau)\,\frac{\partial u}{\partial\tau}\,\mathrm{d}\tau+\underset{\sim}{K}u=f \tag{16}$$

where the operators are identified as follows:

$$\underset{\sim}{M}\triangleq \underset{\sim}{M}_{ij}=\rho(r)\delta_{ij} \tag{17}$$

$$\underset{\sim}{R}\triangleq \underset{\sim}{R}_{ij}=-\sum_{k,l=1}^{3}\frac{\partial}{\partial x_k}\left[G_{ij}^{kl}(r,t)\frac{\partial}{\partial x_l}\right] \tag{18}$$

$$\underset{\sim}{K}\triangleq \underset{\sim}{K}_{ij}=-\sum_{k,l=1}^{3}\frac{\partial}{\partial x_k}\left[C_{ij}^{kl}(r)\frac{\partial}{\partial x_l}\right] \tag{19}$$

and the externally applied force vector f and displacement u are defined in Eqs. (7) and (12). Here, the symbol $\underset{\sim}{R}$ is used to distinguish between the damping operator derived consistently from the constitutive relation and the more commonly used damping operatory symbol $\underset{\sim}{C}$ or $\underset{\sim}{D}$. It is observed from Eqs. (18) and (19) that the form of the damping and stiffness operators is the same. This means that in a spatial discretization procedure such as the finite-element method, the algorithm for developing the stiffness matrix $\boldsymbol{K}$ and the damping matrix $\boldsymbol{R}(t)$ is the same. Therefore, the extension of a dynamical elastic model to a dynamical viscoelastic model is relatively simple, and if a computer code for a particular elastic structure already exists, it need be modified only slightly to account for viscoelastic effects.

From Eq. (18) it is observed that the damping operator $\underset{\sim}{R}$ is time-dependent, unlike the mass and stiffness operators. In addition, the functional relationship is such that the damping operator tends to zero as time increases arbitrarily; this is in keeping with the fading memory property, i.e.,

$$\lim_{t\to\infty}\underset{\sim}{R}(r,t)=\underset{\sim}{0} \tag{20}$$

The response u at time t as obtained from integro-differential Eq. (16) is dependent on the entire history of the response up to time t and can be determined uniquely in conjunction with the initial conditions

$$u(0)=u_0 \text{ and } \dot{u}(0)=v_0 \tag{21}$$

For the undamped problem when

$$G_{ij}^{kl}\equiv 0 \tag{22}$$

the damping operator vanishes,

$$\underset{\sim}{R} \equiv \underset{\sim}{0} \tag{23}$$

as expected, and we obtain the standard undamped differential motion equation

$$\underset{\sim}{M}\ddot{u}^0 + \underset{\sim}{K}u^0 = f \tag{24}$$

Here, superscript $(.)^0$ is used to differentiate between the elastic and viscoelastic representations.

IV. Properties of Operators

The mass, damping, and stiffness operators, Eqs. (17-19), are self-adjoint with respect to space but not with respect to time:

$$(u_1,\underset{\sim}{M}u_2) = (u_2,\underset{\sim}{M}u_1), \qquad (u_1,\underset{\sim}{R}u_2) = (u_2,\underset{\sim}{R}u_1),$$

$$(u_1,\underset{\sim}{K}u_2) = (u_2,\underset{\sim}{K}u_1) \tag{25}$$

In addition, they obey the definiteness properties

$$(u,\underset{\sim}{M}u) > 0, \quad (u,\underset{\sim}{R}u) \geq 0, \quad (u,\underset{\sim}{K}u) \geq 0 \tag{26}$$

for any $u \neq 0$. That is, the mass operator is positive-definite, while the damping and stiffness operators are positive-semidefinite in general; the product $(.\ ,.)$ is given by

$$(u_1,u_2) \triangleq \int_V u_1^T u_2 \, \mathrm{d}V \tag{27}$$

V. Eigenproblem

We wish to study the free vibration problem, in particular, we wish to analyze the decaying oscillatory motion of a viscoelastic structure in the absence of external forces after it has undergone a small perturbation from equilibrium. The response to an arbitrary loading f can then be expanded in terms of the time-independent eigenfunctions. Hence, we seek a formulation of the eigenproblem that transforms the integro-differential operator in Eq. (16) into

$$\lambda \underset{\sim}{A}y = \underset{\sim}{B}y \tag{28}$$

The convolution integral in the motion Eq. (16) suggests the application of the Laplace or Fourier transform. Either is applicable, but we choose the Laplace representation.

Taking the Laplace transform (Laplace variable p) of integro-differential motion Eq. (16) yields

$$[p^2\underset{\sim}{M} + p\underset{\sim}{\tilde{R}}(r,p) + \underset{\sim}{K}]\tilde{u} = \tilde{f} + \underset{\sim}{M}\left[\frac{\partial u(r,0)}{\partial t} + pu(r,0)\right]$$

$$+ \underset{\sim}{\tilde{R}}u(r,0) \tag{29}$$

where

$$\underset{\sim}{\tilde{R}}(r,p) = -\sum_{k,l=1}^{3} \frac{\partial}{\partial x_k}\left[\tilde{G}_{ij}^{kl}(r,p)\frac{\partial}{\partial x_l}\right] \tag{30}$$

$$\tilde{G}_{ij}^{kl}(r,p) = \int_0^\infty \frac{C_{ij}^{kl(*)}(r,\theta)\gamma(\theta)}{p+\theta}\,\mathrm{d}\theta \tag{31}$$

Laplace transformed quantities are denoted by $(\tilde{.})$.

We shall expand the solution for an arbitrary loading f in terms of the eigenfunctions $v(r)$, which are space-dependent but independent of initial conditions. Therefore, we consider the homogeneous counterpart of Eq. (29), i.e., the free vibration of a viscoelastic structure with zero initial conditions:

$$\tilde{f} \equiv 0, \qquad u(r,0) = \frac{\partial}{\partial t}u(r,0) = 0 \tag{32}$$

The eigenproblem so obtained is

$$[p^2\underset{\sim}{M} + p\underset{\sim}{\tilde{R}}(r,p) + \underset{\sim}{K}]\,v = 0 \tag{33}$$

This is a nonstandard eigenproblem; an approximate solution technique using a version of Jacobi's root perturbation method is the subject of another paper. In the analysis to follow, it is convenient to work in first-order form, so the eigenproblem Eq. (33) is expressed as

$$p\underset{\sim}{A}q + \underset{\sim}{\tilde{B}}(p)q = 0 \tag{34}$$

where

$$\underset{\sim}{A} = \begin{matrix} \underset{\sim}{M} & \underset{\sim}{0} \\ \underset{\sim}{0} & -\underset{\sim}{K} \end{matrix} \tag{35}$$

$$\underset{\sim}{\tilde{B}}(p) = \begin{matrix} \underset{\sim}{\tilde{R}}(p) & \underset{\sim}{K} \\ \underset{\sim}{K} & \underset{\sim}{0} \end{matrix} \tag{36}$$

$$q^T = [pv \;\; v] \tag{37}$$

Here, the operators $\underset{\sim}{M}$, $\underset{\sim}{\tilde{R}}$, and $\underset{\sim}{K}$ and consequently $\underset{\sim}{A}$ and $\underset{\sim}{\tilde{B}}$ are functions of the spatial coordinates; for simplicity, this dependence is not noted explicitly in Eqs. (34-37). From Eq. (35), it is observed that the operator $\underset{\sim}{A}$ is real, symmetric, and self-adjoint. That is, for any vectors q_1,q_2,

$$(q_1,\underset{\sim}{A}q_2) = (\underset{\sim}{A}q_1,q_2) \tag{38}$$

Also, the operator $\underset{\sim}{\tilde{B}}$ is complex, symmetric, and self-adjoint:

$$(q_1,\underset{\sim}{\tilde{B}}q_2) = (\underset{\sim}{\tilde{B}}q_1,q_2) \tag{39}$$

It is observed that neither $\underset{\sim}{A}$ nor $\underset{\sim}{\tilde{B}}$ exhibit definiteness properties.

For the undamped problem, when

$$\underset{\sim}{\tilde{R}} \equiv \underset{\sim}{0} \tag{40}$$

the eigenproblem becomes

$$p^0\underset{\sim}{A}q^0 + \underset{\sim}{B}^0q^0 = 0 \tag{41}$$

where the matrix operator $\underset{\sim}{B}^0$ is symmetric as before but is now real. That is,

$$\underset{\sim}{B}^0 = \begin{matrix} \underset{\sim}{0} & \underset{\sim}{K} \\ \underset{\sim}{K} & \underset{\sim}{0} \end{matrix} \tag{42}$$

while

$$q^{0^T} = [p^0v^0 \;\; v^0] \tag{43}$$

The superscript $(.)^0$ is used to distinguish the undamped elastic eigenproblem from the damped viscoelastic problem. Thus, the eigenvalues and eigenfunctions for the undamped problem are denoted by p^0 and q^0. It is noted that the matrix operator $\underset{\sim}{A}$ defined by Eq. (35) does not involve the damping operator $\underset{\sim}{\tilde{R}}(p)$. Hence, the matrix operator $\underset{\sim}{A}^0$ of the undamped problem is identical to the matrix operator $\underset{\sim}{A}$ of the damped problem.

VI. Orthonormality Conditions

Before we derive the orthonormality conditions for the eigenfunctions of the viscoelastic problem, let us first review the classical orthonormality conditions for the eigenfunctions of the undamped eigenproblem, Eq. (41). For the αth mode, we have

$$p_\alpha^0 \underset{\sim}{A} q_\alpha^0 + \underset{\sim}{B}^0 q_\alpha^0 = \mathbf{0}, \qquad \alpha = 1,2,3,\ldots \tag{44}$$

The orthonormality conditions can then be expressed as

$$(q_\alpha^0, \underset{\sim}{A} q_\beta^0) = A_\alpha^0 \delta_{\alpha\beta} \qquad \alpha,\beta = 1,2,\ldots \tag{45}$$

and

$$(q_\alpha^0, \underset{\sim}{B}^0 q_\beta^0) = -p_\alpha^0 A_\alpha^0 \delta_{\alpha\beta} \tag{46}$$

where A_α^0 is chosen as a normalizing factor.

We now turn to the task of finding the orthogonality conditions for the damped viscoelastic problem related to the eigenproblem

$$p_\alpha \underset{\sim}{A} q_\alpha + \tilde{\underset{\sim}{B}}_\alpha q_\alpha = \mathbf{0}, \qquad \alpha = 1,2,3,\ldots \tag{47}$$

For simplicity, the notation

$$\tilde{\underset{\sim}{B}}_\alpha \triangleq \tilde{B}(p_\alpha) \tag{48}$$

is adopted. First, from Eq. (47), it can be observed that the eigenvalues and eigenfunctions for the damped problem occur in complex conjugate pairs, since from Eq. (31),

$$\bar{G}_{ij}^{kl}(r,p) = \bar{G}_{ij}^{kl}(r,\bar{p}) \tag{49}$$

Now, applying $(q_\beta,.)$ to Eq. (47) yields

$$p_\alpha (q_\beta, \underset{\sim}{A} q_\alpha) + (q_\beta, \tilde{\underset{\sim}{B}}_\alpha q_\alpha) = 0 \tag{50}$$

and utilizing the self-adjoint properties of $\underset{\sim}{A}$ and $\tilde{\underset{\sim}{B}}$ from Eqs. (38) and (39), Eq. (50) becomes

$$p_\alpha (q_\alpha, \underset{\sim}{A} q_\beta) + (q_\alpha, \tilde{\underset{\sim}{B}}_\alpha q_\beta) = 0 \tag{51}$$

Interchanging α and β in Eq. (50) gives

$$p_\beta (q_\alpha, \underset{\sim}{A} q_\beta) + (q_\alpha, \tilde{\underset{\sim}{B}}_\beta q_\beta) = 0 \tag{52}$$

and then subtracting Eq. (51) from Eq. (52) yields the orthogonality condition

$$(p_\alpha - p_\beta)(q_\alpha, \underset{\sim}{A} q_\beta) + (q_\alpha, [\tilde{\underset{\sim}{B}}_\alpha - \tilde{\underset{\sim}{B}}_\beta] q_\beta) = 0, \qquad p_\alpha \neq p_\beta \tag{53}$$

Hence, the eigenfunctions are not orthogonal to $\underset{\sim}{A}$, unlike in Eq. (45) for the undamped problem. For the normality condition, we shall take

$$(q_\alpha, \underset{\sim}{A} q_\alpha) = A_\alpha \tag{54}$$

A second orthogonality relation can be obtained by multiplying Eq. (47) by p_β and then taking $(q_\beta,.)$. This gives

$$p_\alpha p_\beta (q_\beta, \underset{\sim}{A} q_\alpha) + p_\beta (q_\beta, \tilde{\underset{\sim}{B}}_\alpha q_\alpha) = 0 \tag{55}$$

and using the symmetry of $\underset{\sim}{A}$ and $\tilde{\underset{\sim}{B}}$ yields

$$p_\alpha p_\beta (q_\alpha, \underset{\sim}{A} q_\beta) + p_\beta (q_\alpha, \tilde{\underset{\sim}{B}}_\alpha q_\beta) = 0 \tag{56}$$

Interchanging the indices α and β in Eq. (55) produces

$$p_\alpha p_\beta (q_\alpha, \underset{\sim}{A} q_\beta) + p_\alpha (q_\alpha, \tilde{\underset{\sim}{B}}_\beta q_\beta) = 0 \tag{57}$$

Upon subtracting Eq. (56) from Eq. (57), the second orthogonality condition is obtained as

$$(q_\alpha, [p_\beta \tilde{\underset{\sim}{B}}_\alpha - p_\alpha \tilde{\underset{\sim}{B}}_\beta] q_\beta) = 0, \qquad p_\alpha \neq p_\beta \tag{58}$$

Equations (53) and (58) are two orthogonality relations for the viscoelastic problem. The former relation, Eq. (53), is important in the derivation of modal identities, which is the subject of discussion of the next section.

VII. Modal Identities

It has been a traditional practice of dynamicists to analyze a flexible structure in terms of its eigenvalues and eigenfunctions. Hughes[8] has shown that two integrals of the eigenfunctions $\boldsymbol{P}_\alpha$ and $\boldsymbol{H}_\alpha$, which correspond to the linear and angular momenta of mode α, satisfy many identities. These identities provide a guide for establishing the importance of various modes and a numerical check on the completeness of a reduced (truncated) model. Several ideas put forth by Likins et al.[9] and Hughes and Skelton[10] are available for mode selection, and the ones that rely solely on the structural dynamics are those that depend on the eigenvalues p_α and modal coefficients $\boldsymbol{P}_\alpha$ and $\boldsymbol{H}_\alpha$.

We now extend Hughes' elastic results to encompass the case of a constrained viscoelastic structure, as shown in Fig. 1. The structure is constrained at point 0 to eliminate rigid-body motion. First, we shall find an expansion of the inverse of the mass, damping, and stiffness operators in terms of the eigenmodes. The identities for the modal coefficients $\boldsymbol{P}_\alpha$ and $\boldsymbol{H}_\alpha$ are then established using the expansion of $(\underset{\sim}{M}^{-1}, \underset{\sim}{K}^{-1}, \underset{\sim}{0})$.

The constraint condition makes the stiffness operator $\underset{\sim}{K}$ positive-definite; thus, the inverse of operator $\underset{\sim}{A}$ [Eq. (35)] exists. Expanding $\underset{\sim}{A}^{-1}$ in terms of its eigenfunctions,

$$\underset{\sim}{A}^{-1} \triangleq \sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta} q_\alpha)(q_\beta \tag{59}$$

where the notation for the product .)(. is defined explicitly by

$$\underset{\sim}{A}^{-1} x \triangleq \sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta} q_\alpha (q_\beta, x) \tag{60}$$

and $c_{\alpha\beta}$ are unknown coefficients to be determined. Now, for Eq. (59) to be a valid expansion, $\underset{\sim}{A}^{-1}$ must satisfy the eigenproblem

$$p_\gamma q_\gamma + \underset{\sim}{A}^{-1} \tilde{\underset{\sim}{B}}_\gamma q_\gamma = \mathbf{0} \tag{61}$$

Thus, substituting for $\underset{\sim}{A}^{-1}$ from Eq. (59) into Eq. (61) and using Eq. (50) gives

$$p_\gamma \left[q_\gamma - \sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta} (q_\beta, \underset{\sim}{A} q_\gamma) q_\alpha \right] = \mathbf{0}, \qquad \gamma = 1,2,3,\ldots \tag{62}$$

For a constrained structure

$$p_\gamma \neq 0, \qquad \gamma = 1,2,3,\ldots \tag{63}$$

and since q_α are eigenfunctions, from Eq. (62) we obtain

$$\sum_{\beta=1}^{\infty} c_{\alpha\beta} (q_\beta, \underset{\sim}{A} q_\gamma) = \delta_{\alpha\gamma}, \qquad \alpha,\gamma = 1,2,3,\ldots \tag{64}$$

or in matrix form we can write

$$\boldsymbol{CD} = \mathbf{1} \tag{65}$$

where the elements of the matrices are identified as follows:

$$C = c_{\alpha\beta} \tag{66}$$

$$D = (q_\beta, \underset{\sim}{A} q_\gamma) \tag{67}$$

$$\mathbf{1} = \delta_{\alpha\gamma} \tag{68}$$

Equation (65) is sufficient to solve for the unknown coefficients $c_{\alpha\beta}$, provided D is nonsingular. It should be noted that matrix D as defined in Eq. (67) is not diagonal, since the eigenfunctions of a viscoelastic structure are not orthogonal to $\underset{\sim}{A}$, as is evident from Eq. (53). This is unlike the elastic problem, where the orthogonality condition Eq. (45) diagonalizes D.

Three more expansions from $\underset{\sim}{A}^{-1}$ can be obtained. For this, we first rewrite q in terms of v using Eq. (37). The expansion for $\underset{\sim}{A}^{-1}$ then becomes

$$\underset{\sim}{A}^{-1} = \sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta} \begin{matrix} p_\alpha p_\beta v_\alpha)(v_\beta & p_\alpha v_\alpha)(v_\beta \\ p_\beta v_\alpha)(v_\beta & v_\alpha)(v_\beta \end{matrix} \tag{69}$$

Also, From Eq. (35), we have

$$\underset{\sim}{A}^{-1} = \begin{matrix} \underset{\sim}{M}^{-1} & \underset{\sim}{0} \\ \underset{\sim}{0} & -\underset{\sim}{K}^{-1} \end{matrix} \tag{70}$$

The operator $\underset{\sim}{K}$ is invertible, since the structure is constrained. So comparing the two expressions for $\underset{\sim}{A}^{-1}$, Eqs. (69) and (70), we obtain

$$\underset{\sim}{M}^{-1} = \sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta} p_\alpha p_\beta v_\alpha)(v_\beta \tag{71}$$

$$\underset{\sim}{K}^{-1} = -\sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta} v_\alpha)(v_\beta \tag{72}$$

$$\underset{\sim}{0} = \sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta} p_\alpha v_\alpha)(v_\beta \tag{73}$$

The preceding expressions can be used to establish identities in terms of the modal coefficients, which are identified as follows. First, consider the reaction force and torque at the constrained point 0 to the viscoelastic body (Fig. 1);

$$F_R(t) = -\int_V y(r,t)\,\mathrm{d}V \tag{74}$$

$$T_R(t) = -\int_V r^x y(r,t)\,\mathrm{d}V \tag{75}$$

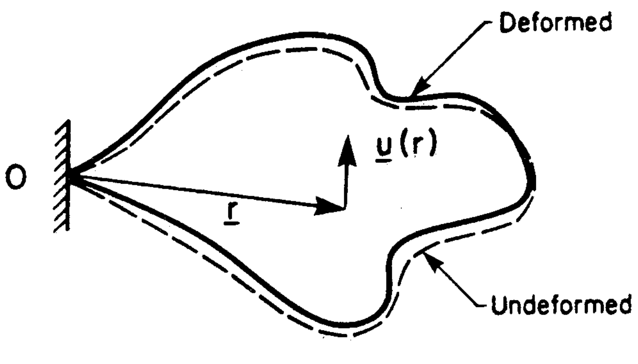

Fig. 1 Constrained viscoelastic body.

where

$$y \triangleq \underset{\sim}{K} u = f - \underset{\sim}{M} \ddot{u} - \int_0^t \underset{\sim}{R}(r,t-\tau)\frac{\partial u}{\partial \tau}\,\mathrm{d}\tau \tag{76}$$

and $(.)^x$ is the skew-symmetric 3×3 matrix associated with the cross product. The displacement function under our assumption of linearity can be taken to be separable in the spatial coordinates and the variable t. So

$$u(r,t) = \sum_{\alpha=1}^{\infty} v_\alpha(r) v_\alpha(t) \tag{77}$$

where $v_\alpha(r)$ and $v_\alpha(t)$ are in general complex. The reaction force and torque can then be written as

$$F_R(t) = -F_E(t) + \sum_{\alpha=1}^{\infty}\left[P_\alpha \ddot{v}_\alpha + \int_0^t R_\alpha(t-\tau)\frac{\mathrm{d}v_\alpha}{\mathrm{d}\tau}\,\mathrm{d}\tau\right] \tag{78}$$

$$T_R(t) = -T_E(t) + \sum_{\alpha=1}^{\infty}\left[H_\alpha \ddot{v}_\alpha + \int_0^t Q_\alpha(t-\tau)\frac{\mathrm{d}v_\alpha}{\mathrm{d}\tau}\,\mathrm{d}\tau\right] \tag{79}$$

where

$$F_E(t) \triangleq \int_V f(r,t)\,\mathrm{d}V \tag{80}$$

$$T_E(t) \triangleq \int_V r^x f(r,t)\,\mathrm{d}V \tag{81}$$

$$P_\alpha \triangleq \int_V \rho(r) v_\alpha(r)\,\mathrm{d}V$$

$$= (\rho\mathbf{1}, v_\alpha) \tag{82}$$

$$H_\alpha \triangleq \int_V \rho(r) r^x v_\alpha(r)\,\mathrm{d}V$$

$$= -(\rho r^x, v_\alpha) \tag{83}$$

$$R_\alpha(t) \triangleq \int_V \underset{\sim}{R}(r,t) v_\alpha(r)\,\mathrm{d}V$$

$$= (\underset{\sim}{R}, v_\alpha) \tag{84}$$

$$Q_\alpha(t) \triangleq \int_V r^x \underset{\sim}{R}(r,t) v_\alpha(r)\,\mathrm{d}V$$

$$= -(\underset{\sim}{R} r^x, v_\alpha) \tag{85}$$

The constants P_α and H_α are termed the modal momentum and modal angular momentum coefficients, since the momentum and angular momentum of the viscoelastic body are

$$\sum_{\alpha=1}^{\infty} P_\alpha \dot{v}_\alpha \quad \text{and} \quad \sum_{\alpha=1}^{\infty} H_\alpha \dot{v}_\alpha$$

respectively. Coefficients $R_\alpha(t)$ and $Q_\alpha(t)$ are identified as the time-dependent modal linear and angular coefficients associated with the damping operator $\underset{\sim}{R}$.

The modal identities can now be set up in terms of the modal coefficients. Using Eq. (17), we have

$$(\rho\mathbf{1}, \underset{\sim}{M}^{-1}\rho\mathbf{1}) = m\mathbf{1} \tag{86}$$

where m is the total mass of the viscoelastic body. If the expansion for $\underset{\sim}{M}^{-1}$ as given by Eq. (71) is used, we obtain

$$(\rho\mathbf{1},\underset{\sim}{M}^{-1}\rho\mathbf{1}) = \sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta}p_\alpha p_\beta(\rho\mathbf{1},v_\alpha)(v_\beta,\rho\mathbf{1})$$
$$= \sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta}p_\alpha p_\beta \boldsymbol{P}_\alpha \boldsymbol{P}_\beta^T \tag{87}$$

Comparison of Eqs. (86) and (87) yields the matrix identity

$$\sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta}p_\alpha p_\beta \boldsymbol{P}_\alpha \boldsymbol{P}_\beta^T = m\mathbf{1} \tag{88}$$

It should be noted that the double sum in Eq. (88) is real, since the coefficients $c_{\alpha\beta}$, the eigenvalues, the eigenfunctions, and thus the modal coefficients all appear in complex conjugate pairs.

Similarly, two more identities involving the expansion for $\underset{\sim}{M}^{-1}$ can be established. These are

$$\sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta}p_\alpha p_\beta \boldsymbol{H}_\alpha \boldsymbol{H}_\beta^T = \boldsymbol{J} \tag{89}$$

$$\sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta}p_\alpha p_\beta \boldsymbol{H}_\alpha \boldsymbol{P}_\beta^T = \boldsymbol{c}^x \tag{90}$$

where $\boldsymbol{c}$ and $\boldsymbol{J}$ are first and second moments of inertia, respectively, and

$$\boldsymbol{c}^x(t) \triangleq -(\rho \boldsymbol{r}^x,\underset{\sim}{M}^{-1}\rho\mathbf{1}) \tag{91}$$

$$\boldsymbol{J} \triangleq (\rho \boldsymbol{r}^x,\underset{\sim}{M}^{-1}\rho \boldsymbol{r}^x) \tag{92}$$

Proceeding in the manner outlined, using Eq. (72) for operator $\underset{\sim}{K}^{-1}$, we can establish three more identities. For this, we require the flexibility operator, which can be expressed as an integral operator

$$\underset{\sim}{F}\,[f(\boldsymbol{r})] \triangleq \underset{\sim}{K}^{-1}\,[f(\boldsymbol{r})] = \int_V F(\boldsymbol{r},\xi)f(\xi)\mathrm{d}V \tag{93}$$

where $\boldsymbol{F}$ is symmetric and positive-definite, making $\boldsymbol{F}$ symmetric and positive-definite. Therefore,

$$\sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta}\boldsymbol{P}_\alpha \boldsymbol{P}_\beta^T = -\int_V\int_V \rho(\boldsymbol{r})\boldsymbol{F}(\boldsymbol{r},\xi)\rho(\xi)\mathrm{d}V\mathrm{d}V \tag{94}$$

$$\sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta}\boldsymbol{H}_\alpha \boldsymbol{H}_\beta^T = -\int_V\int_V \rho(\boldsymbol{r})\boldsymbol{r}^x\boldsymbol{F}(\boldsymbol{r},\xi)\rho(\xi)\xi^x\mathrm{d}V\mathrm{d}V \tag{95}$$

$$\sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta}\boldsymbol{H}_\alpha \boldsymbol{P}_\beta^T = -\int_V\int_V \rho(\boldsymbol{r})\boldsymbol{r}^x\boldsymbol{F}(\boldsymbol{r},\xi)\rho(\xi)\mathrm{d}V\mathrm{d}V \tag{96}$$

Using Eq. (73) for the null operator, we obtain

$$\sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta}p_\beta \boldsymbol{P}_\alpha \boldsymbol{P}_\beta^T = \mathbf{0} \tag{97}$$

$$\sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta}p_\beta \boldsymbol{H}_\alpha \boldsymbol{H}_\beta^T = \mathbf{0} \tag{98}$$

$$\sum_{\alpha,\beta=1}^{\infty} c_{\alpha\beta}p_\beta \boldsymbol{H}_\alpha \boldsymbol{P}_\beta^T = \mathbf{0} \tag{99}$$

This completes the derivation of the modal identities, Eqs. (88–90), (94–96), and (97–99). These can be used as a numerical check on the solution of the eigenproblem or, more importantly, for mode selection, as shown in Refs. 9 and 10.

The modal identities for the elastic (undamped) problem can be shown to involve only a single sum. The eigenfunctions $\boldsymbol{q}^0$ are orthogonal to $\underset{\sim}{A}$, as seen from Eq. (45), so matrix $\boldsymbol{D}$ in Eq. (67) becomes

$$\boldsymbol{D}^0 = (\boldsymbol{q}_\beta^0,\underset{\sim}{A}\boldsymbol{q}_\gamma^0)$$
$$= A_\beta^0\delta_{\beta\gamma} \tag{100}$$

From Eq. (65) we obtain

$$c_{\alpha\beta} = A_\beta^{0-1}\delta_{\alpha\beta} \tag{101}$$

So the double sum in the modal identities derived earlier reduces to a single sum, i.e., Eq. (88) becomes

$$\sum_{\alpha=1}^{\infty} A_\alpha^{0-1}p_\alpha^0 p_\alpha^0 \boldsymbol{P}_\alpha^0 \boldsymbol{P}_\alpha^{0T} = m\mathbf{1} \tag{102}$$

In keeping with the notation used earlier, superscript $(.)^0$ is used for all quantities (normalizing factor A_α^0, eigenvalues p_α^0, and modal coefficient $\boldsymbol{P}_\alpha^0$) of the elastic problem. Similarly, the double sum in Eqs. (89–90), (94–96), and (97–99) reduces to a single sum (see Hughes[8]).

VIII. Example—Cantilevered Viscoelastic Beam

This section illustrates the concepts discussed earlier with the solution of a simple one-dimensional problem: a cantilevered viscoelastic beam in pure flexure. The stress-strain relation for the one-dimensional problem is

$$\sigma = \int_0^t G(t-\tau)\frac{\partial\epsilon}{\partial\tau}\,\mathrm{d}\tau + C\epsilon$$
$$G(t) = \int_0^\infty C^{(*)}(\theta)\gamma(\theta)e^{-\theta t}\mathrm{d}\theta \tag{103}$$

Using this in the principle of virtual work in the absence of external forces, the equation of motion for the viscoelastic beam is

$$\rho\ddot{w} + I\left[\int_0^t G(t-\tau)\frac{\partial w''''}{\partial\tau}\,\mathrm{d}\tau + Cw''''\right] = 0 \tag{104}$$

where ρ is the mass/length and I is the second moment of the cross-sectional area. To formulate the eigenproblem, we first take the Laplace transform, which gives

$$p^2\rho\tilde{w} + I[p\tilde{G}(p) + C]\frac{\partial^4\tilde{w}}{\partial x^4} = \rho\left[\frac{\partial w(0)}{\partial t} + pw(0)\right] + \tilde{G}w(0)$$
$$\tilde{G}(p) = \int_0^\infty \frac{C^{(*)}(\theta)\gamma(\theta)}{p+\theta}\,\mathrm{d}\theta \tag{105}$$

The corresponding eigenproblem is therefore

$$p^2\rho\phi + I[p\tilde{G}(p) + C]\phi'''' = 0 \tag{106}$$

and the eigenmodes ϕ_j are obtained from the differential equation

$$\phi_j'''' - z_j^4 \rho \phi_j = 0, \qquad z_j - \text{separation constant}$$
$$j = 1, 2, \ldots \tag{107}$$

and satisfy the orthonormality condition

$$\int_0^l \phi_i(x)\phi_j(x)\mathrm{d}x = 1 \tag{108}$$

The eigenmodes that satisfy Eqs. (107) and (108) are

$$\phi_j(x) = \frac{1}{l^{1/2}}[(\cos z_j x - \cosh z_j x) - \alpha_j(\sin z_j x - \sinh z_j x)] \tag{109}$$

where

$$z_j = \frac{\epsilon_j}{l}, \qquad \epsilon_j^4 = \frac{\omega_j^2 l^4}{CI} \tag{110}$$

$$\alpha_j = \frac{\cos\epsilon_j + \cosh\epsilon_j}{\sin\epsilon_j + \sinh\epsilon_j} \tag{111}$$

and ϵ_j are the roots of the transcendental equation

$$\cosh\epsilon_j \cos\epsilon_j + 1 = 0 \tag{112}$$

The integro-differential equation for the modal coordinate f_j is

$$\ddot{f}_j + \omega_j^2 [g_j(t) + f_j] = 0 \tag{113}$$

with

$$g_j(t) \overset{\Delta}{=} \frac{1}{C}\int_0^t G(t-\tau)\frac{\mathrm{d}f_j}{\mathrm{d}\tau}\,\mathrm{d}\tau \tag{114}$$

The deflection $w(x,t)$ can then be obtained from

$$w(x,t) = \sum_{j=1}^{\infty} \phi_j(x) f_j(t) \tag{115}$$

The reaction force and torque at the fixed end of the beam can be obtained from

$$F_R(t) = \sum_{j=1}^{\infty} [P_j \ddot{f}_j + CIz_j^4 P_j g_j(t)] \tag{116}$$

$$T_R(t) = \sum_{j=1}^{\infty} [H_j \ddot{f}_j + CIz_j^4 H_j g_j(t)] \tag{117}$$

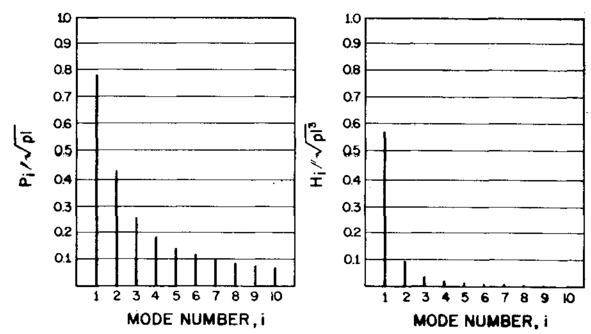

Fig. 2 Nondimensionalized modal coefficents.

Here, the damping force

$$\left[= CI \sum_{j=1}^{\infty} z_j^4 P_j g_j(t) \right]$$

is a linear combination of the momentum coefficients P_j. Using Eq. (109), the modal momentum coefficients for the beam, as given by Eqs. (82) and (83), become

$$P_i = \rho\int_0^l \phi_i d(x)\mathrm{d}x$$
$$= 2(\rho l)^{1/2}\frac{\alpha_i}{\epsilon_i} \tag{118}$$

$$H_i = \rho\int_0^l x\phi_i(x)\mathrm{d}x$$
$$= \frac{2(\rho l^3)^{1/2}}{\epsilon_i^2} \tag{119}$$

As shown in Fig. 2, the modal parameters P_i and H_i decrease monotonically with mode number i, and H_i decrease faster than P_i. Therefore, identities involving parameter H_i converge faster than the ones involving P_i. This result was shown by Hughes[11] for the elastic problem, i.e., when $G(t) \equiv 0$. Thus, for this simple problem, model order reduction amounts to modal truncation, i.e., retaining the first N modes and discarding the rest. However, in general, P_i and H_i do not decrease nonotonically with mode number i, so a reduced-order model involves a process of modal selection based on P_i and H_i.

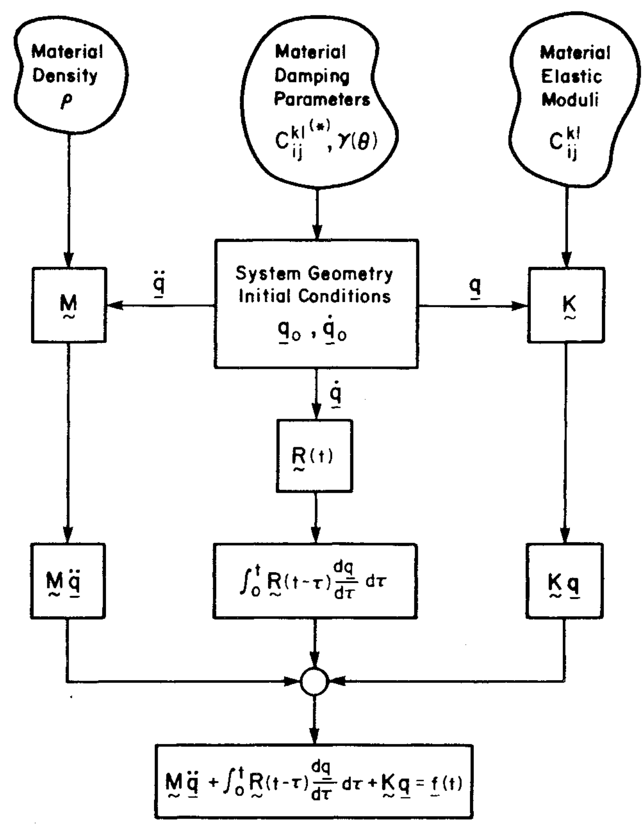

Fig. 3 Flow chart for a dynamical viscoelastic model.

IX. Closing Remarks

In this paper, a general dynamical model for material damping based on viscoelastic assumptions is introduced. As observed from the flowchart for the viscoelastic model depicted in Fig. 3, the motion equations are no longer differential but rather integro-differential in nature. All operators (mass, damping, and stiffness) are completely defined and arise consistently in the formulation.

The eigenproblem is formulated by taking the Laplace transform of the equation of motion. The eigenmodes of the problem are shown to satisfy certain orthogonality conditions that have been derived in this paper. Furthermore, it is noted that the desired viscoelastic eigenproblem is nonstandard; the solution of this problem using a perturbation analysis is the subject of another paper. In this regard, the accuracy of a numerical solution of the eigenproblem can be checked by the modal identities derived in this paper. The only aspect remaining to be determined is the viscous moduli and decay distribution function $\gamma(\theta)$. The viscous moduli can be determined from a creep test as shown by Smith[12] for an orthotropic graphite-reinforced epoxy. As for the decay distribution function $\gamma(\theta)$, one approach to determining it could involve correlating the analytical and experimental response (displacement) for a simple structure, such as a beam. Once the distribution is established, a consistent damping study can be incorporated in a dynamical model at the design stage.

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